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CENTRALLY SYMMETRIC CONVEX BODIES AND DISTRIBUTIONS II

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ABSTRACT

In continuation of a previous work we study the generating distributions of centrally symmetric convex bodies and obtain some more geometric formulas and new characterizations of zonoids and generalized zonoids.

Introduction

This is a continuation of the article [11], where we had assigned an even distribution T_K, the so-called *generating distribution*, to each centrally symmetric convex body K in E^d ($d \geqq 2$). Moreover, in [11] we studied the behaviour of these distributions, found formulas for their connection with the geometric properties of K, and derived a characterization of zonoids.

In section 2 of this paper we shall establish some results concerning the projection functions of K. Using a result of [12] on surface area measures, we will generalize in section 3 our characterization of zonoids. In a similar way we derive a characterization of generalized zonoids (as defined in [10]). We start with some remarks concerning [11].

Symbols and notations are taken from [11] and the results of [11] are presupposed.

1. Remarks on [11]

As P. R. Goodey pointed out to me, there is a shorter proof of theorem 5.2 in [11]. One direction follows from theorem 5.1, the other from the fact that the line segment s with endpoints u and $-u$, $u \in \Omega$, is a zonoid and

$$v(L_i, u) = \frac{d}{2} V(L_i, \cdots, L_i, s), \qquad i = 1, 2.$$

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Secondly, by a misprint and an uncomplete proof, theorem 4.5 as it stands is not correct. The restriction $T|_A$ of the distribution T to the open set A is an element of $\mathcal{D}'(A)$, not of $\mathcal{D}'(\Omega)$. But then, in the formulation of the theorem, it is not clear whether and how

$$T_K|_{A^\perp}(|\langle u,\cdot\rangle|)$$

is defined. We therefore state theorem 4.5 in a slightly different form, for which the proof given in [11] is correct and which preserves the essential idea.

THEOREM 1.1. *Let $K \in \mathcal{K}$ and let $A \subset \Omega$ be an open, connected subset. For any $K' \in \mathcal{K}$ with $T_K|_{A^\perp} = T_{K'}|_{A^\perp}$ there exists a $v \in E^d$ such that*

$$H_K(u) = H_{K'}(u) + \langle u, v\rangle$$

for all $u \in A$.

If $T_K = \rho_K \in \mathcal{M}(\Omega)$, the original version of theorem 4.5 is correct, because the restriction $\rho_K|_{A^\perp}$ is again in $\mathcal{M}(\Omega)$. However, in this case, we can give a short proof by putting

$$v = 2\int_{A^*} x\,\rho_K(dx),$$

where

$$A^* = \{x \in \Omega\backslash A^\perp | \langle x, u\rangle > 0 \quad \text{for all } u \in A\}.$$

Finally, two minor corrections should be mentioned. On pp. 364–366 the index n appears several times and should be replaced by d. In the formula

$$V(K, L, \cdots, L) = \frac{1}{d}\int_\Omega H_K(u)\mu_{d-1}(L; du)$$

used in the proofs of theorem 5.1 and theorem 5.2 the factor $1/d$ had been omitted.

There are some additional references to be mentioned which are not cited in [10] or [11]. Lindquist gave in [4] the proof of a result, announced in his paper cited in [10]. In [5] he gives a partial solution to the problem of characterizing the set of generating measures of generalized zonoids. His solution is different from the one we gave in [10]. Moreover, in [5] a formula for the mixed volume is obtained which is just our theorem 4 in [10]. Matheron [6] studied zonoids in connection with random set theory and Poisson networks. He derived formulas for the projection functions, the quermassintegrals, and other special mixed volumes. Some of the formulas coincide with those found in [10], others are generalized in the next section. With a recent paper of Goodey [3] we are concerned in the last section.

2. Projection formulas

For the following we have to introduce additional notations.

Let $j \in \{1, \cdots, d-1\}$ and let $u_1, \cdots, u_{d-j} \in \Omega$ be linearly independent. Then, let $E(u_1, \cdots, u_{d-j})$ be the $(d-j)$-dimensional subspace of E^d spanned by $u_1, \cdots, u_{d-j}$, and let $E^\perp(u_1, \cdots, u_{d-j})$ be the orthogonal space. For convex bodies $K_1, \cdots, K_j$ let $v(K_1, \cdots, K_j; u_1, \cdots, u_{d-j})$ be the j-dimensional mixed volume of the projection of $K_1, \cdots, K_j$ onto $E^\perp(u_1, \cdots, u_{d-j})$. $v(K_1, \cdots, K_j; u_1, \cdots, u_{d-j})$ is the mixed projection function studied in [9].

For two j-spaces $E, F \subset E^d$ let $|E, F|$ be the absolute value of the determinant of the projection from E onto F. We have $|E, F| = |F, E| = |E^\perp, F^\perp|$.

Expressions like $v(K_1, \cdots, K_j; u_1, \cdots, u_{d-j})$ or $|F, E(u_1, \cdots, u_{d-j})|$ etc. are defined to be zero for linearly dependent $(d-j)$-tuples $(u_1, \cdots, u_{d-j})$.

We start with an auxiliary result which is easy to prove.

Proposition 2.1. *We have*

$$D_d(x_1, \cdots, x_d) = D_j(x_1, \cdots, x_j) D_{d-j}(x_{j+1}, \cdots, x_d) \cdot$$

$$\cdot |E(x_1, \cdots, x_j), E^\perp(x_{j+1}, \cdots, x_d)|.$$

Next we will show how the generating distributions and the mixed volumes of projections are connected.

Theorem 2.2. *For $j \in \{1, \cdots, d-1\}$, $K_1, \cdots, K_j \in \mathcal{K}$ and $u_1, \cdots, u_{d-j} \in \Omega$ we have*

$$v(K_1, \cdots, K_j; u_1, \cdots, u_{d-j})$$

$$= \frac{2^j}{j!} (T_{K_1} \otimes \cdots \otimes T_{K_j})(D_j \cdot |E(., \cdots, .), E^\perp(u_1, \cdots, u_{d-j})|).$$

Proof. It is obviously sufficient to prove the theorem for $K_1 = \cdots = K_j = K$ and K a generalized zonoid. The general case follows then by approximation (compare the remark on p. 361 of [11]) and by multilinearity of the mixed volume on the left side and of the tensor product on the right side of the above formula.

Thus,

$$H_K(u)=\int_\Omega |\langle x,u\rangle|\,\rho_K(dx),\qquad u\in E^d,$$

and (see [1], p. 46, and [2], p. 19), for linearly independent $u_1,\cdots,u_{d-j}$

$$v(\underbrace{K,\cdots,K}_{j};u_1,\cdots,u_{d-j})=\frac{\binom{d}{d-j}}{\kappa_{d-j}}\,V(\underbrace{K,\cdots,K}_{j},\underbrace{B_{d-j},\cdots,B_{d-j}}_{d-j})$$

where B_{d-j} is the $(d-j)$-dimensional unit ball in $E(u_1,\cdots,u_{d-j})$.

It follows from theorem 4.2 in [11] and Proposition 2.1 that

$$\begin{aligned}
&v(\underbrace{K,\cdots,K}_{j};u_1,\cdots,u_{d-j})\\
&=\frac{\binom{d}{d-j}}{\kappa_{d-j}}\cdot\frac{2^d}{d!}\int_\Omega\cdots\int_\Omega D_d(x_1,\cdots,x_d)\rho_K(dx_1)\cdots\rho_K(dx_j)\cdot\\
&\quad\cdot\rho_{B_{d-j}}(dx_{j+1})\cdots\rho_{B_{d-j}}(dx_d)\\
&=\frac{2^d}{\kappa_{d-j}\cdot j!\,(d-j)!}\int_\Omega\cdots\int_\Omega\Bigg[\int_{\Omega\cap E(u_1,\cdots,u_{d-j})}\cdots\int_{\Omega\cap E(u_1,\cdots,u_{d-j})}\\
&\quad D_{d-j}(x_{j+1},\cdots,x_d)\cdot|E(x_{j+1},\cdots,x_d),E^\perp(x_1,\cdots,x_j)|\cdot\\
&\quad\cdot\rho_{B_{d-j}}(dx_{j+1})\cdots\rho_{B_{d-j}}(dx_d)\Bigg]\cdot D_j(x_1,\cdots,x_j)\rho_K(dx_1)\cdots\rho_K(dx_j).
\end{aligned}$$

Here we have used that $\rho_{B_{d-j}}$ is concentrated on $\Omega\cap E(u_1,\cdots,u_{d-j})$.

Because of

$$|E(x_{j+1},\cdots,x_d),E^\perp(x_1,\cdots,x_j)|=|E(x_1,\cdots,x_j),E^\perp(u_1,\cdots,u_{d-j})|$$

for all linearly independent $x_{j+1},\cdots,x_d\in\Omega\cap E(u_1,\cdots,u_{d-j})$ and

$$\int_{\Omega\cap E(u_1,\cdots,u_{d-j})}\cdots\int_{\Omega\cap E(u_1,\cdots,u_{d-j})}D_{d-j}(x_{j+1},\cdots,x_d)\cdot$$

$$\cdot\rho_{B_{d-j}}(dx_{j+1})\cdots\rho_{B_{d-j}}(dx_d)=\frac{(d-j)!}{2^{d-j}}\cdot\kappa_{d-j}$$

we obtain

$$\begin{aligned}
&v(\underbrace{K,\cdots,K}_{j};u_1,\cdots,u_{d-j})\\
&=\frac{2^j}{j!}\int_\Omega\cdots\int_\Omega D_j(x_1,\cdots,x_j)|E(x_1,\cdots,x_j),E^\perp(u_1,\cdots,u_{d-j})|\cdot\\
&\quad\cdot\rho_K(dx_1)\cdots\rho_K(dx_j).
\end{aligned}$$

Q.E.D.

As a consequence, Theorem 2.2 implies

THEOREM 2.3. *For* $j \in \{1, \cdots, d-1\}$ *and* $K_1, \cdots, K_d \in \mathcal{K}$ *we have*

$$V(K_1, \cdots, K_d) = \frac{2^{d-j} j!}{d!} (T_{K_{j+1}} \otimes \cdots \otimes T_{K_d})(v(K_1, \cdots, K_j; ., \cdots, .) \cdot D_{d-j}).$$

PROOF. Using Theorem 2.2, Proposition 2.1 and theorem 4.2 of [11], we get

$$\begin{aligned} V(K_1, \cdots, K_d) &= \frac{2^d}{d!} (T_{K_1} \otimes \cdots \otimes T_{K_d})(D_d) \\ &= \frac{2^{d-j} j!}{d!} (T_{K_{j+1}} \otimes \cdots \otimes T_{K_d})_{(x_{j+1}, \cdots, x_d)} \Big[\frac{2^j}{j!} D_{d-j}(x_{j+1}, \cdots, x_d) \cdot \\ &\qquad \cdot (T_{K_1} \otimes \cdots \otimes T_{K_j})(D_j \cdot | E(., \cdots, .), E^{\perp}(x_{j+1}, \cdots, x_d)|) \Big] \\ &= \frac{2^{d-j} j!}{d!} (T_{K_{j+1}} \otimes \cdots \otimes T_{K_d})(D_{d-j} \cdot v(K_1, \cdots, K_j; ., \cdots, .)). \end{aligned}$$

Q.E.D.

For zonoids, Theorems 2.2 and 2.3 have been proved with a slightly different method by Matheron ([6], p. 103).

For later use we put

$$v_k(K; u_1, \cdots, u_{d-j}) = v(\underbrace{K, \cdots, K}_{k}, \underbrace{B, \cdots, B}_{j-k}; u_1, \cdots, u_{d-j})$$

where $j \in \{1, \cdots, d-1\}$ and $k \in \{0, \cdots, j\}$.

3. Zonoids and generalized zonoids

In theorem 5.1 of [11] we characterized zonoids in the set $\mathcal{K}$ by inequalities between mixed volumes. Recently Goodey [3] proved the following more general version of theorem 5.1 and a similar characterization of generalized zonoids.

(3.1) *A convex body* K *is a zonoid if and only if*

$$V(K, L_1, \cdots, L_1) \leqq V(K, L_2, \cdots, L_2)$$

for all convex bodies L_1, L_2 *which fulfill* $v(L_1, u) \leqq v(L_2, u)$ *for all* $u \in \Omega$.

(3.2) *A convex body* K *is a generalized zonoid if and only if*

$$|V(K,L_1,\cdots,L_1)-V(K,L_2,\cdots,L_2)|\leqq c(K)\cdot\sup_{u\in\Omega}|v(L_1,u)-v(L_2,u)|$$

for all convex bodies L_1, L_2 *with a suitable constant* $c(K)$ *independent of* L_1, L_2.

The symmetry of K which is not assumed in both statements is a consequence of the following fact.

(3.3) *A convex body* K *is centrally symmetric if and only if*

$$V(K,L,\cdots,L)=V(K,L^*,\cdots,L^*)$$

for all convex bodies L.

Here $L^*=\{x \mid -x\in L\}$.

(3.3) is a special case of a proposition of Schneider ([7], pp. 226–227).

The one direction of (3.1) and (3.2) follows as in the proof of theorem 5.1 in [11]. The converse of (3.1) is a consequence of (3.3) and our theorem 5.1 in [11]. In the proof of the latter we used a well-known property of distributions, namely,

$$T_K\in\mathcal{M}_+(\Omega) \text{ if and only if } T_K(f)\geqq 0 \text{ for all } f\in\mathcal{D}(\Omega), f\geqq 0,$$

and the fact that for each $f\in\mathcal{D}(\Omega)$ there exist $L_1,L_2\in\mathcal{K}$ with $f(u)=v(L_2,u)-v(L_1,u), u\in\Omega$. This follows because of $\mathcal{D}(\Omega)\subset\mathcal{E}(\Omega)$ and $\mathcal{S}^*_{d-1}-\mathcal{S}^*_{d-1}=\mathcal{M}(\Omega)$, where $\mathcal{S}^*_{d-1}$ is the set of $(d-1)$-th order surface area measures of bodies $K\in\mathcal{K}$.

The converse of (3.2) could be proved analogously because of the following property of distributions ([8], p. 25):

$$T_K\in\mathcal{M}(\Omega) \text{ if and only if } |T_K(f)|\leqq c(K)\sup_{u\in\Omega}|f(u)|$$

$$\text{for all } f\in\mathcal{D}(\Omega).$$

In order to obtain the announced generalizations we replace the mixed volumes $V(K,L_i,\cdots,L_i)$ by

$$V(K,\underbrace{L_i,\cdots,L_i}_{j},\underbrace{B,\cdots,B}_{d-1-j})$$

and the projection functions $v(L_i,u)=v_{d-1}(L_i;u)$ by $v_j(L_i;u)$, $j\in\{1,\cdots,d-1\}$. Therefore, instead of $\mathcal{S}^*_{d-1}$, the corresponding set $\mathcal{S}^*_j$ of j-th order surface area measures of bodies $K\in\mathcal{K}$ must be considered. But for $\mathcal{S}^*_j$ with $j<d-1$ it is no longer true that $\mathcal{S}^*_j-\mathcal{S}^*_j=\mathcal{M}(\Omega)$ (see [12]). Nevertheless, we can accomplish the proofs by using the following result of [12]:

(3.4) *$\mathcal{S}_j^* - \mathcal{S}_j^*$ is dense in $\mathcal{M}(\Omega)$.*

We start with the following generalization of (3.3).

PROPOSITION 3.5. *Let $j \in \{1, \cdots, d-1\}$, $k \in \{1, \cdots, d-j\}$, and let K be a convex body of dimension at least $j+1$ if $j > 1$ and arbitrary if $j = 1$. Then K is centrally symmetric if and only if*

$$V(\underbrace{K, \cdots, K}_{j}, \underbrace{L, \cdots, L}_{k}, \underbrace{B, \cdots, B}_{d-k-j}) = V(\underbrace{K, \cdots, K}_{j}, \underbrace{L^*, \cdots, L^*}_{k}, \underbrace{B, \cdots, B}_{d-k-j})$$

for all convex bodies L.

PROOF. If K is centrally symmetric, the assertion is obvious.

The converse is again a special case of the proposition of Schneider [7] if $k = 1$. The general case could be easily reduced to the case $k = 1$ in the following way.

From

$$V(\underbrace{K, \cdots, K}_{j}, \underbrace{L, \cdots, L}_{k}, \underbrace{B, \cdots, B}_{d-k-j}) = V(\underbrace{K, \cdots, K}_{j}, \underbrace{L^*, \cdots, L^*}_{k}, \underbrace{B, \cdots, B}_{d-k-j})$$

for all L, especially for $L = \alpha M + B$, $\alpha \geqq 0$, where M is a convex body, we get

$$\sum_{i=0}^{k} \alpha^i \binom{k}{i} V(\underbrace{K, \cdots, K}_{j}, \underbrace{M, \cdots, M}_{i}, \underbrace{B, \cdots, B}_{d-i-j})$$

$$= \sum_{i=0}^{k} \alpha^i \binom{k}{i} V(\underbrace{K, \cdots, K}_{j}, \underbrace{M^*, \cdots, M^*}_{i}, \underbrace{B, \cdots, B}_{d-i-j})$$

for all $\alpha \geqq 0$ and all M. Hence,

$$V(\underbrace{K, \cdots, K}_{j}, \underbrace{M, \cdots, M}_{i}, \underbrace{B, \cdots, B}_{d-i-j}) = V(\underbrace{K, \cdots, K}_{j}, \underbrace{M^*, \cdots, M^*}_{i}, \underbrace{B, \cdots, B}_{d-i-j})$$

for all $i \in \{0, \cdots, k\}$ and all M. Q.E.D.

Next, we prove a more general characterization of zonoids.

THEOREM 3.6. *Let $j \in \{1, \cdots, d-1\}$ and let K be a convex body. K is a zonoid if and only if*

$$V(K,\underbrace{L_1,\cdots,L_1}_{j},\underbrace{B,\cdots,B}_{d-1-j})\leqq V(K,\underbrace{L_2,\cdots,L_2}_{j},\underbrace{B,\cdots,B}_{d-1-j})$$

for all convex bodies L_1, L_2 *which fulfill* $v_j(L_1;u)\leqq v_j(L_2;u)$ *for all* $u\in\Omega$.

Proof. Suppose, first, that K is a zonoid and belongs to $\mathscr{K}$, that is

$$H_K(u)=\int_\Omega |\langle x,u\rangle|\rho_K(dx),\qquad \rho_K\in\mathscr{M}_+(\Omega).$$

Then, by Theorem 2.3, we have for $i=1,2$

$$V(K,\underbrace{L_i,\cdots,L_i}_{j},\underbrace{B,\cdots,B}_{d-1-j})=\frac{2}{d}\int_\Omega v_j(L_i;u)\rho_K(du).$$

Therefore, $v_j(L_1;u)\leqq v_j(L_2;u)$ for all $u\in\Omega$ implies

$$V(K,\underbrace{L_1,\cdots,L_1}_{j},\underbrace{B,\cdots,B}_{d-1-j})\leqq V(K,\underbrace{L_2,\cdots,L_2}_{j},\underbrace{B,\cdots,B}_{d-1-j}).$$

On the other hand, suppose this inequality is valid for all L_1, L_2 with $v_j(L_1;.)\leqq v_j(L_2;.)$. Then, because of $v_j(L;.)=v_j(L^*;.)$, we get

$$V(K,\underbrace{L,\cdots,L}_{j},\underbrace{B,\cdots,B}_{d-1-j})=V(K,\underbrace{L^*,\cdots,L^*}_{j},\underbrace{B,\cdots,B}_{d-1-j})$$

for all L, from which the symmetry of K follows by Proposition 3.5. After a translation, K belongs to $\mathscr{K}$, hence T_K exists and is in $\mathscr{M}_+(\Omega)$ if $T_K(f)\geqq 0$ for all $f\in\mathscr{D}(\Omega)$, $f\geqq 0$.

But $\mathscr{S}_j^*-\mathscr{S}_j^*$ is dense in $\mathscr{M}(\Omega)$ ((3.4)), thus

$$\left\{f\in\mathscr{C}(\Omega)\,\middle|\, f(u)=\int_\Omega|\langle x,u\rangle|\rho(dx),\ \rho\in\mathscr{S}_j^*-\mathscr{S}_j^*\right\}$$

is dense in $\mathscr{C}(\Omega)$, which implies that

$$\{v_j(L_2;.)-v_j(L_1;.)\,|\,L_1,L_2\in\mathscr{K}\}\cap\mathscr{D}(\Omega)$$

is dense in $\mathscr{D}(\Omega)$. Therefore $T_K\geqq 0$ if $T_K(v_j(L_1;.))\leqq T_K(v_j(L_2;.))$ for $L_1,L_2\in\mathscr{K}$ with $v_j(L_i;.)\in\mathscr{D}(\Omega)$ and $v_j(L_1;.)\leqq v_j(L_2;.)$. But

$$T_K(v_j(L_i;.)) = \frac{d}{2} V(K, \underbrace{L_i, \cdots, L_i}_{j}, \underbrace{B, \cdots, B}_{d-1-j})$$

(Theorem 2.3) and the theorem is proved. Q.E.D.

The following theorem gives the corresponding characterization of generalized zonoids.

THEOREM 3.7. *Let $j \in \{1, \cdots, d-1\}$ and let K be a convex body. K is a generalized zonoid if and only if*

$$\Big| V(K, \underbrace{L_1, \cdots, L_1}_{j}, \underbrace{B, \cdots, B}_{d-1-j}) - V(K, \underbrace{L_2, \cdots, L_2}_{j}, \underbrace{B, \cdots, B}_{d-1-j}) \Big|$$

$$\leqq c(K) \cdot \sup_{u \in \Omega} | v_j(L_1; u) - v_j(L_2; u) |$$

for all convex bodies L_1, L_2 with a suitable constant $c(K)$ independent of L_1, L_2.

PROOF. The proof is similar to that of Theorem 3.6. For one direction, we have

$$\Big| V(K, \underbrace{L_1, \cdots, L_1}_{j}, \underbrace{B, \cdots, B}_{d-1-j}) - V(K, \underbrace{L_2, \cdots, L_2}_{j}, \underbrace{B, \cdots, B}_{d-1-j}) \Big|$$

$$= \frac{2}{d} \Big| \int_\Omega (v_j(L_1; u) - v_j(L_2; u)) \rho_K(du) \Big|.$$

$$\leqq \frac{2}{d} \| \rho_K \| \sup_{u \in \Omega} | v_j(L_1; u) - v_j(L_2; u) |.$$

For the converse we observe the already mentioned fact, that $T_K \in \mathcal{M}(\Omega)$ if $|T_K(f)| \leqq c(K) \sup_{u \in \Omega} |f(u)|$ for all $f \in \mathcal{D}(\Omega)$. Q.E.D.

In view of Theorem 2.3 one might guess that a more general characterization of zonoids is possible, perhaps as bodies K which have the following property $\mathcal{P}(j, k)$.

$\mathcal{P}(j,k)$: *For $j \in \{1, \cdots, d-1\}$ and $k \in \{1, \cdots, d-j\}$ we have*

$$V(\underbrace{K, \cdots, K}_{j}, \underbrace{L_1, \cdots, L_1}_{k}, \underbrace{B, \cdots, B}_{d-k-j}) \leqq V(\underbrace{K, \cdots, K}_{j}, \underbrace{L_2, \cdots, L_2}_{k}, \underbrace{B, \cdots, B}_{d-k-j})$$

for all L_1, L_2 with $v_k(L_1; u_1, \cdots, u_j) \leqq v_k(L_2; u_1, \cdots, u_j)$ for all $u_1, \cdots, u_j \in \Omega$.

Property $\mathcal{P}(1,k)$ characterizes zonoids by Theorem 3.7, but property $\mathcal{P}(d-1,1)$ obviously characterizes the set of centrally symmetric convex bodies in the set of convex bodies of dimension d. Furthermore, each zonoid K has property $\mathcal{P}(j,k)$ for arbitrary j,k, because of (Theorem 2.3)

$$V(\underbrace{K,\cdots,K}_{j},\underbrace{L_i,\cdots,L_i}_{k},\underbrace{B,\cdots,B}_{d-k-j})$$

$$=\frac{2^j(d-j)!}{d!}\int_\Omega\cdots\int_\Omega v_k(L_i;u_1,\cdots,u_j)D_j(u_1,\cdots,u_j)\cdot$$

$$\cdot\rho_K(du_1)\cdots\rho_K(du_j),\qquad i=1,2.$$

On the other hand, each K which has property $\mathcal{P}(j,k)$ and dimension at least $j+1$ is centrally symmetric in view of Proposition 3.5.

Hence, property $\mathcal{P}(j,k)$ characterizes a class of convex bodies between the class of zonoids and the class of centrally symmetric bodies; a class which, as we conjecture, should be independent of k. It would be of interest to find another characterization of these bodies for $j\in\{2,\cdots,d-2\}$.

For an interpretation of such problems in terms of orderings on the set of convex bodies see the remarks in [11], pp. 365–366.

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